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Small Spaces with Large Point Character

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Let (X, ϱ) be a metric space. An *open covering* $\mathscr{U}$ of (X, ϱ) is a family of open subsets of X with $X = \bigcup\mathscr{U}$. The $\mathscr{U}$ is called *uniform* if there exists $\varepsilon > 0$ such that for every $x \in X$ there is $U \in \mathscr{U}$ which contains the ε-ball $B_\varepsilon(X) = \{y, \varrho(X, y) < \varepsilon\}$.

By the well-known theorem of A. H. Stone [10] each metric space is paracompact and hence each open covering $\mathscr{U}$ of (X, ϱ) has an open locally *finite refinement* $\mathscr{V}$—i.e. there is an open covering $\mathscr{U}$ with the two following properties:

(a) for each $x \in X$ there is a neighborhood of x which meets only finitely many members of $\mathscr{V}$;

(b) for every $V \in \mathscr{V}$ there is an $U \in \mathscr{U}$ with $V \subset U$.

The question, whether in Stone's theorem open coverings may be replaced by uniform ones (i.e. whether every uniform covering has a locally finite uniform refinement) was originally formulated by A. H. Stone [9] and is mentioned also in Isbell's book [3]. The answer to this question is clearly positive for any Euclidean space or more generally separable space. However, the question was answered negatively in [4] and [8], where it was shown that the space $l_\infty(\kappa)$ (κ infinite and sufficiently large, Pelant's bound $\kappa \geqslant 2^{\aleph_1}$ is better than that of [8]) does not have the above property. In [7] we proved that the space $l_p(\kappa)$, $0 < p < \infty$ (κ large) does not have this property either. This together with some other related questions was further studied in [6]. All those examples have large number of points. The aim of this note is to construct a relatively small space (smallest possible) giving a negative answer to Stone's question. We will need the following definitions: We say that a covering $\mathscr{U}$ is c-bounded if diam $U < c$ for every $U \in \mathscr{U}$. Here diam $U = \sup\{\varrho(x, y), x, y \in \mathscr{U}\}$ we say that a covering $\mathscr{U}$ is bounded if it is c-bounded for some $c > 0$. The central notion of our paper is the notion of *point character*:

DEFINITION. Let κ be a cardinal number, we say that the point character of a metric space (X, ϱ) is bigger than κ and we denote it by $pc(X, \varrho) > \kappa$ if there exists $c > 0$ such that for every c-bounded uniform covering $\mathscr{U}$ of X there is a point $x \in X$ which is contained in at least κ members of $\mathscr{U}$. We set $pc(X, \varrho) = \lambda$ if $\lambda = \min\{K; pc(X, \varrho) \not> \kappa\}$. A space with $pc(X, \varrho) \leqslant \aleph_0$ is also called *point finite*. As clearly, for any Euclidean space E_n $pc(E_n) = n + 2$ the point character provides suitable generalization of the notion of dimension for the 'infinite dimensional' case.

The Stone question is now equivalent to ask whether every metric space (X, ϱ) is point finite. (Clearly, the positive answer for the former would give a positive answer to the latter and one can show (see [3]) that the reversed implication holds as well.)

It has been observed in [5], that the point character of a metric space cannot be larger than its cardinality and thus $pc(X, \varrho) \geqslant \kappa$ implies $|X| \geqslant \kappa$. On the other hand the results proved in [4], [6], [7], [8] do not ensure the existence of the space with the same cardinality and dimension (point character). The above mentioned examples, giving negative answer to Stone's question have point character $\aleph_1$ but their cardinality is larger. Here we prove the following

THEOREM 0.1. *There exists a metric space* (X, ϱ) *with*

$$pc(X, \varrho) = |X| = \aleph_1$$

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This is related to a recent result of J. Pelant (oral communication), who proved by different means that $pc(l_0(\omega)) \geqslant \aleph_1$. Thus assuming the continuum hypothesis $l_\infty(\omega)$ provides such an example also. Our method is based on the theory of graphs.

By a *graph* G we shall understand a couple (V, E), where $V = V(G)$ is the set of *vertices* and $E = E(G) \subset [V]^2$ is the set of *edges*—$[V]^2$ is the set of all unordered pairs of distinct elements of V. The *chromatic number* $\chi(G)$ of a graph G is the least number of colors, needed to color the vertices of G so that no two adjacent vertices get the same color. Any such coloring is called a *chromatic coloring*. We say that the vertices $v_1, v_2, \ldots, v_n$ form a *cycle* of G if $\{v_i, v_{i+1}\} \in E(G)$ for $i = 1, 2, \ldots, n-1$ and $\{v_1 v_n\} \in E(G)$. If x is a vertex of G the *neighborhood* $N_G(x)$ of x in G is the set of all vertices adjacent to x. If x and y are two vertices of G, *the distance of x and y in G* ($\mathrm{dist}_G(x, y)$) is the minimal positive integer n such that there exists a path with vertices $x = x_0, x_1, \ldots, x_n = y$ (i.e. $\{x_i, x_{i+1}\} \in E(G)$) for $i = 0, 1, \ldots, n-1$. If such sequence does not exist we put $\mathrm{dist}_G(x, y) = \infty$. Let (V_1, F_1) (V_2, F_2) be two graphs, by *a homomorphism* we understand a mapping ϕ: $V_1 \to V_2$ such that $\{v, w\} \in F_1$ implies $\{\phi(v), \phi(w)\} \in F_2$. It is a matter of routine to verify $\chi(V_1, F_1) \leqslant \chi(V_2, F_2)$.

Graphs and Point Character

We shall need the following.

1.1. Definition. Let G be a graph, j a positive integer and κ a cardinal. We say that the jth point character of G is bigger than κ – $pc^j G > \kappa$ if for every coloring ϕ of vertices of G with the property:

$$\mathrm{dist}_G(x, y) \geqslant j \text{ implies } \phi(x) \neq \phi(y),$$

there exists a vertex x_0 of G the neighborhood of which is colored with at least κ colors.

1.2. Theorem. *Let G_j, $j = 1, 2, \ldots$ be a sequence of connected graphs on disjoint sets of vertices with $pc^j G_j > \kappa$. If we denote by (X, ϱ) the metric space defined by $X = \bigcup_{j=1}^{\infty} V(G_j)$ with metric*

$$\varrho(x, y) = \begin{cases} (1/j)\, \mathrm{dist}_{G_j}(x, y), & \text{for } x, y \in V(G_j), \\ \infty, & \text{for } x \in V(G_i),\ y \in V(G_j),\ i \neq j, \end{cases}$$

then the point character of the space (X, ϱ) is bigger than κ—i.e. $pc(X, \varrho) > \kappa$.

Proof. Take a 1-bounded uniform covering $\mathscr{U}$ of X. Thus for $U \in \mathscr{U}$, diam $U < 1$ and moreover, there exists $\varepsilon > 0$ such that for every $x \in X_\varrho$ $B_\varepsilon(x) = \{y, \varrho(x, y) < \varepsilon\} \subset U$ for some $U \in \mathscr{U}$. To every $x \in X$ choose $U \in \mathscr{U}$ with this property. This defines the mapping ϕ: $\bigcup_{j \in \omega} V(G_j) \to \mathscr{U}$.

Take j_0 so large that $1/j_0 < \varepsilon$ and consider a mapping ϕ restricted to the vertex set of the graph G_{j_0}. Evidently, this mapping satisfies

$$\mathrm{dist}_{G_{j_0}}(x, y) \geqslant j_0 \text{ implies } \phi(x) \neq \phi(y).$$

Thus, there exists a vertex $x_0 \in V(G_{j_0})$ the neighborhood of which is colored by at least κ colors. But $\mathrm{dist}_{G_{j_0}}(x_0, y) = 1$ (i.e. $y \in N_{G_{j_0}}(x_0)$) implies $\varrho(x_0, y) < \varepsilon$ and thus x_0 is contained in at least κ members of $\mathscr{U}$.

ERDÖS–HAJNAL GRAPHS

We shall use the following result of Erdös and Hajnal (cf. [2]).

2.1. THEOREM. *For every infinite cardinal number κ and positive integer n there exists a graph $G_n(\kappa)$ with the cardinality of the vertex set κ chromatic number κ and which does not contain odd cycles of length* $3, 5, \ldots, 2n + 1$.

First we prove one auxiliary lemma. Let $H = (V, F)$ be a graph and $<$ a linear ordering of its vertices. For every vertex $v \in V$ set

$$L_H(v) = \{w: w < v \text{ and } \{w, v\} \in F\}.$$

2.2. LEMMA. *Let λ be an infinite cardinal and let $H = (V, F)$ be a graph with $|V| \leqslant 2^\lambda$. Let $<$ be a linear ordering of V. Let ϕ be a chromatic coloring of V with the property that $L_H(v)$ is monochromatic for every $v \in V$. Then $\chi(H) \leqslant \lambda$.*

PROOF. We may think about colors as ordinals smaller than 2^λ. To every vertex $v \in V$ assign a pair $\psi(v) = (\alpha_v, \beta_v)$ where α_v is color of $L_H(v)$ (if $L_H(v) \neq \theta$ then set $\alpha_v = 0$) and β_v is the color of v—i.e. $\beta_v = \phi(v)$. Then $\{v, w\} \in F$ implies $\alpha_v = \beta_w$ or $\alpha_w = \beta_v$ and thus ψ defines a homomorphism from (V, F) into a graph with vertex set $2^\lambda \times 2^\lambda$ (as usual, we identify 2^λ with the set of ordinals smaller than 2^λ) and with vertices (α, β), (γ, δ) joined by an edge iff $\beta = \gamma$ or $\alpha = \delta$. It was shown in [1] that the chromatic number of this graph is λ and thus $\chi(H) \leqslant \lambda$. We shall use further the following:

CLAIM. *Let (V, E) be a graph with $\chi(V, E) = \aleph_1$ and let $E = \bigcup_{k=1}^{i} E_k$ be a partition into finitely many parts, then there exists $k_0 \leqslant i$ such that $\chi(V, E_{k_0}) = \aleph_1$.*

For $\kappa = \aleph_1$ let $G_n(\aleph_1) = G_n$ be the graph from 2.1. Theorem. We prove:

2.3. LEMMA. *$pc^j G_n > \aleph_0$ for every $n \geqslant j$.*

PROOF. Let ϕ be a coloring of the graph G_n with the property described in 1.1. Definition. Suppose that the neighborhood of each vertex $v \in V(G_n)$ has only finitely many colors. Let $V = V(G_n) = \bigcup_{i=1}^{\infty} C_i$ be a partition, where $C_i = \{x \in V(G_n);\ N_G(x)$ is colored by i colors$\}$. As $\chi(G_n) = \aleph_1$, there exists a positive integer i such that the subgraph G^* of G induced on a set G_i has the chromatic number $\aleph_1$. Clearly, every vertex v of $G^* = (V, E)$ has a neighborhood $N_{G^*}(v)$ colored by at most i colors. Fix now a linear ordering of the vertices of G^* and suppose moreover that the set $\{\phi(v);\ v \in V(G^*)\}$ is also linearly ordered. Set $\phi[L_{G^*}(v)] = \{\phi(w);\ w \in L_{G^*}(v)\}$ and define a partition $E(G^*) = \bigcup_{k=1}^{i} E_k$ by $E_k = \{\{v, w\};\ w \in L_{G^*}(v)$ and $\phi(w)$ is the kth largest color in $\phi[L_{G^*}(v)]\}$. Then, clearly there exists $k_0 \leqslant i$ such that $\chi(B) = \aleph_1$, where $B = (V(G^*), E_{k_0})$. Moreover $L_B(v)$ is monochromatic for every $v \in V(G^*)$. Consider the partition $E_{k_0} = F_1 \cup F_2$ defined by

$$F_1 = \{\{x, y\};\ \phi(x) \neq \phi(y)\},$$

$$F_2 = \{\{x, y\};\ \phi(x) = \phi(y)\}.$$

As ϕ restricted to the graph $H(V(G^*), F_1)$ is a chromatic coloring, we have according to 2.2., Lemma $\chi(H) \leqslant \aleph_0$. Thus $\chi(V(G^*), F_2) = \aleph_1$ and $(V(G^*), F_2)$ contains an odd cycle with length necessarily bigger than $2n + 1 \geqslant 2j + 1$. Hence there exist two vertices $x, y \in V(G^*)$ with $\operatorname{dist}_{G_n}(x, y) > j$, but $\varrho(x) = \varrho(y)$ contradiction.

Main Result

Proof of Theorem 0.1. Combining 1.2. Theorem and 2.2. Lemma we get an example of a metric space (X, ϱ) with $pc(X, \varrho) = |X| = \aleph_1$.

Note that our proof yields the following stronger statement as well:

3.1. Theorem. If $\aleph_\alpha^{\aleph_\beta} < \aleph_{\alpha+1}$ *for every* $\beta < \alpha$, *then there exists a metric space* $(X_{\alpha+1}, \varrho_{\alpha+1})$ *with*

$$pc(X_{\alpha+1}, \varrho_{\alpha+1}) = |X_{\alpha+1}| = \aleph_{\alpha+1}.$$

Theorem 3.1 can be further extended. Recently we proved that the construction of our paper applied directly to generalized Specker graphs (cf. [2]) yields for an arbitrary infinite cardinal κ a metric space (X, ϱ) satisfying

$$pc(X, \varrho) = |X| = \kappa.$$

References

1. P. Erdös and A. Hajnal; On chromatic number of infinite graphs, in *Theory of Graphs* (P. Erdös and G. Katona, eds), Akademiai Kiadó, Budapest, 1968, pp. 83–98.
2. P. Erdös, F. Galvin and A. Hajnal, On set-systems having large chromatic number and not containing prescribed subsystems, in *Infinite and Finite Sets* (A. Hajnal, R. Rado and V. T. Sós, eds), North Holland, Amsterdam, 1976, pp. 425–513.
3. J. R. Isbell, *Uniform Spaces*, Mathematical Surveys 12, AMS, Providence, 1964.
4. J. Pelant, Cardinal reflections and point character of uniformities, in Seminar Uniform Spaces 1973–74 (directed by Z. Frolik), *Mat. ústav ČSAV, Prague* (1975), 149–158.
5. J. Pelant, Universal metric spaces, in Seminar Uniform Spaces 1975–76 (directed by Z. Frolik), *Mat. ústav ČSAV, Prague* (1977), 49–53.
6. J. Pelant and V. Rödl, On coverings of infinite-dimensional metric spaces (submitted for publication in *General Topology*).
7. V. Rödl, Canonical partition relations and point-character of l_p-spaces, in Seminar Uniform Spaces 1976–77 (directed by Z. Frolik), *Mat. ústav ČSAV, Prague* (1978), 79–82.
8. E. V. Shepin, On a problem of Ibell, *Dokl, Akad, Nauk, SSSR* **222** (1975), 541–543.
9. A. H. Stone, Universal space for some metrizable uniformities, *Quart. J. Math.* **11** (1960), 105–115.
10. A. H. Stone, Paracompactness and product spaces, *Bull. AMS* **54** (1948), 977–982.

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